when approached by sampling methods. On the other hand, and in all fairness, it should be admitted that the problem of determining the distribution of general orbits (a topic of importance in assessing the probability of "hitting" a specific "window" in position-velocity space) probably can be handled best by Monte Carlo methods. A good rule to follow is to allow the intrinsic characteristics of the particular problem to dictate the method of treatment to be employed.‡

3. Some Points of Confusion

Contrary to a statement in the preceding comment, it is not assumed in Ref. 1 that β, r, and v are normally distributed with covariance matrix Σ; but it can be inferred, from assumptions made regarding the fundamental guidance parameters, that these quantities possess, approximately, a trivariate normal distribution, with covariance matrix $A\Sigma_x A'$, where the prime denotes the transpose of a matrix. Here, $A = (a_{ij})$ is the $3 \times n$ matrix associated with Eqs. (3) and (4) of Ref. 1, and Σ_x is the covariance matrix of the "errors" associated with the n fundamental guidance parameters.

The x of the preceding note is a random vector of three components, which are distributed independently and identically according to the standard normal law; it is not to be confused with the x of Ref. 1.

4. Monte Carlo vs Analytic Treatment

Two ways of employing the Monte Carlo method in this situation suggest themselves. One is to generate the random n-tuple $(X_1, \ldots, X_n)$ of "errors" in the fundamental guidance parameters, and through Eqs. (3), or equivalently, through Eqs. (6) and (5) of Ref. 1 to simulate (1), the square of the orbital eccentricity. [In this connection the writer is grieved to call attention to the absence of a sign of equality between the f term and the $\sin^2$ term in formula (1) of Ref. 1.] The other method, favored in the preceding note and possibly more efficient from the sampling viewpoint, is to draw random triples from the trivariate normal population with means zero and covariance matrix $A\Sigma_x A'$.

Either method necessitates the computation of the elements of A and Σ_x, which is possibly the most formidable task in the exercise.

The analytic procedure proposed in Ref. 1 was not represented as being superior to any other legitimate method. The computation of the eigenvalues of matrix $(c_{ij}\sigma_i\sigma_j)$, as advocated in Ref. 1, may not be a particularly difficult task, however. The issue depends on the rank of this matrix; if it is $r < n$, then it is well known that $n - r$ of the eigenvalues are zero. Numerical methods for computing the eigenvalues of a real, symmetric matrix are well known and generally available (cf. Ref. 2). Routines for calculating the eignenvalues of a positive-definite (or positive-semidefinite) matrix according to decreasing numerical magnitude are in common use in *factor analysis* studies; they would appear to be especially suitable in the present application where eigenvalues smaller than a preassigned positive constant could be ignored, thus simplifying the analytic treatment by a certain amount.

5. Generation of Random Numbers

The validity of Monte Carlo applications, in a precise mathematical sense, depends on the intrinsic quality of the "random" number sequences employed. The sequences with which we are compelled, for reasons of economy, to work are not random, but "pseudo-random," in the sense that they possess at least some of the properties of random sequences. The particular properties demanded in different applications vary.

A fundamental and commonly required property, moreover, one which is essential in the application recommended in the preceding note, is that the sequence of numbers be *equidistributed*, as defined in Ref. 3. The currently favored mixed congruential methods for generating pseudo-random sequences[4] are the computer implementation of what, in full-precision arithmetic, are called *multiply sequences* in Ref. 3.

Theorem 20 in Ref. 3 establishes that r-dimensional derived sequences (obtained from a multiply sequence) cannot be equidistributed in the r-dimensional unit cube, for any $r > 1$. This result, valid for full-precision arithmetic, casts grave doubts on the implicit postulate that computer-generated sequences possess this desirable property, without which random ordered r-tuples cannot be produced.

The effect of the foregoing is that computed quantities that depend on these r-tuples, such as orbital eccentricity or its square, cannot be considered to be random observations. Consequently, one really has little a priori grounds for believing in the validity of standard statistical procedures when applied to Monte Carlo experiments and no obvious direct method for testing for such validity. As an example, presumed 90% confidence limits for an important system parameter may in fact have an associated true confidence coefficient considerably smaller than 0.9.

‡ The writer, in fact, advocated and employed the Monte Carlo method in treating the original version of the problem of Ref. 1, but did not consider the application worthy of publication.

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Errata: "Optimal Variable-Thrust Transfer of a Power-Limited Rocket between Neighboring Circular Orbits"

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[AIAA J. 2, 339–343 (1964)]

IN the above article, the equations for the out-of-plane motion are incorrect and should be replaced by the following:

$$\frac{C_5}{C_3} = \frac{\tan\tau_f + (w_f/z_f)}{1 - (w_f/z_f)\tan\tau_f} \tag{31}$$

$$C_3 = \frac{(\sin\tau_f + \tau_f\cos\tau_f)z_f - (\tau_f\sin\tau_f)w_f}{\tau_f^2 - \sin^2\tau_f} \tag{40}$$

$$A_z = 2C_3\left\{\left[\frac{\tan\tau_f + (w_f/z_f)}{1 - (w_f/z_f)\tan\tau_f}\right]\cos\tau - \sin\tau\right\} \tag{44}$$

$$\frac{J}{w_0^3 r_0^2} = \frac{(y_f/r_0)^2(5\tau_f + 3\sin\tau_f)}{8[\tau_f(5\tau_f + 3\sin\tau_f) - 16(1 - \cos\tau_f)]} + \frac{i^2}{\tau_f + |\sin\tau_f|} \tag{45}$$

$$\frac{z_f}{r_0 i} = \left(\frac{1 \pm \cos\tau_f}{2}\right)^{1/2} \tag{49}$$

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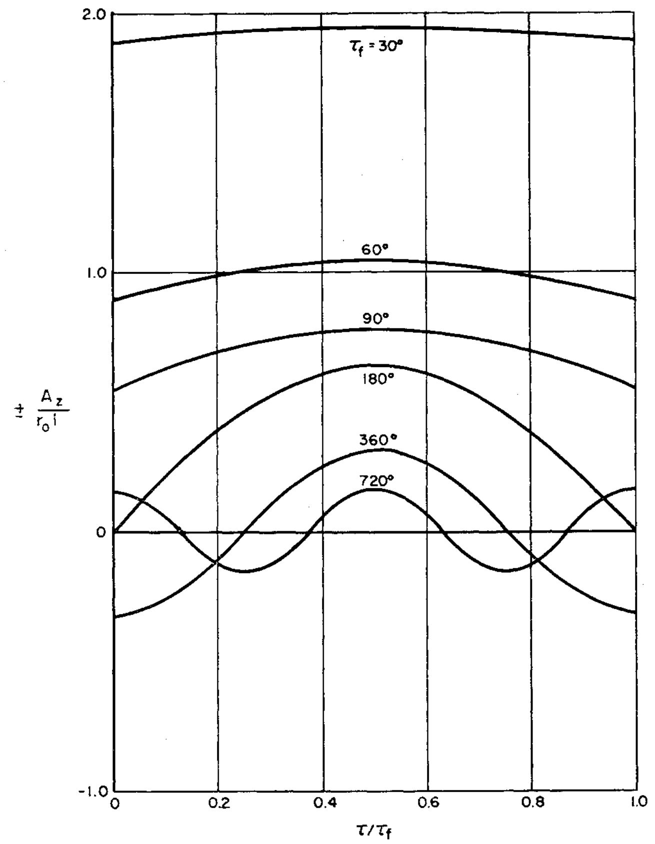

Fig. 5 Normal component of acceleration.

$$\frac{J_2}{i^2} = \frac{1}{\tau_f + |\sin\tau_f|} \tag{51}$$

These changes result in a slightly different graphical form of the optimal out-of-plane component of thrust acceleration $A_z/r_0 i$ shown in Fig. 5. It is noted that Eq. (49) is double-valued, leading to a double-valued steering program $\pm A_z/r_0 i$ in Fig. 5. The resultant change in the payoff J is imperceptible for the scale chosen in Fig. 6 of the article.

Longitudinal Mode Instability

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In recent years, the problems associated with the occurrence of the longitudinal mode of combustion pressure oscillations in liquid rocket motors have been considered solved.[1–3] The solution that has been regarded as being the most complete is that published in Refs. 1 and 3. That solution is based upon the concept of a sensitive time lag.[3] The definitions and discussions of the sensitive time lag are described in detail[1, 3, 4] and will not be discussed herein, but the phase relationships will be briefly reviewed.

By considering a disturbance passing through the combustion zone, it is evident that the sensitive time lag is at a minimum when the pressure is at a maximum for the assumed relationship between them.[4] Thus, for an oscillating pressure, the sensitive time lag will also oscillate and be out of phase. Furthermore, this oscillation in the sensitive time lag can be related to the oscillation in the burning rate, which in turn can be related to an oscillation in the effect of the burning rate. Therefore, if the oscillation in the effect of the burning rate is in phase with the oscillating combustion pressure, the system tends to be unstable. Conversely, for an out-of-phase relationship the system tends to be stable.

By the proper use of these phase relationships for different length combustion chambers,[1] it can be shown that there exists (theoretically) the so-called lower critical length, below which oscillations are damped, as well as the so-called upper critical length, above which the oscillations are also damped. Both of these lengths are predicted by the time lag theory,[3] and it is correctly stated in the conclusions[3] that this upper limit cannot be explained by any other mechanism advanced to date. It is further stated[5] that the results of Ref. 3 show that it is not true that only a lower critical length exists, but that for each mode of longitudinal oscillation there exists a range of lengths outside which the operation is stable, whereas inside it is unstable. In essence then, the validity of the preceding time lag theory[1] is based upon the physical existence of the so-called upper critical length, and the comparison of experiment to theory[1] requires that both lengths exist (see, for example, Figs. 10 and 13 of Ref. 1).

More recent experimental work,[6] however, agrees with earlier experimental work[2, 7] in that no upper critical length was found. In addition, the results of a theory recently completed[8] indicate that no upper critical length should exist. That theory is based upon assuming an Arrhenius-type rate function for the combustion zone dynamics as contrasted with the time lag assumption.[4]

On the basis of the preceding experimental work[2, 6, 7] as well as the preceding theoretical work,[8] there is a serious question concerning the validity of the time lag theory[4] for describing combustion pressure oscillations of the longitudinal mode. It would seem that the necessary existence of the upper critical length clearly invalidates the time lag theory, particularly in those situations where the chemical kinetics predominate, as they do in the case of the premixed gas rocket.[2] That is, the time lag theory is invalid for those situations in which no upper critical length can be found experimentally. In those situations where an upper critical length was found, its existence may possibly be explained by the dissipative effects assumed negligible by the time lag theory.[7] Dissipative effects are also neglected in Ref. 8.

Furthermore, the validity of the time lag theory for describing the transverse mode can also be questioned, at least in those cases for which the chemical kinetics again predominate, that is, at least for the premixed gas motor.[9]

It can be concluded that the solution to the problem of combustion instability does not exist at the present time, but that new theories based upon more realistic assumptions,[8] which agree more closely with the experimental work,[2, 6, 10] make the problems less formidable.

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